Appendix A

Two Dimensional Interaction Integral

In this section, we describe the domain form of the interaction integral (Yau, Wang, and Corten, 1980), (Shih and Asaro, 1988), for the extraction of mixed-mode stress intensity factors. The coordinates are taken to be the local crack tip coordinates with the x_1 axis parallel to the crack faces. For general mixed-mode problems we have the following relationship between the value of the J-integral and the stress intensity factors

$$J = \frac{K_I^2}{E^*} + \frac{K_{II}^2}{E^*} \tag{A.1}$$

where E^* is defined in terms of material parameters E (Young's modulus) and ν (poisson's ratio) as

$$E^* = \begin{cases} E & \text{for plane stress} \\ \frac{E}{1-\nu^2} & \text{for plane strain} \end{cases} \tag{A.2}$$

Two states of a cracked body are considered. State 1, $\left(\sigma_{ij}^{(1)}, \epsilon_{ij}^{(1)}, u_i^{(1)}\right)$, corresponds to the present state and state 2, $\left(\sigma_{ij}^{(2)}, \epsilon_{ij}^{(2)}, u_i^{(2)}\right)$, is an auxiliary state which will be chosen as the asymptotic fields for Mode I or Mode II. The J-integral for the sum of the two states is

$$J^{(1+2)} = \int_\Gamma \left[\frac{1}{2}(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)})(\epsilon_{ij}^{(1)} + \epsilon_{ij}^{(2)})\delta_{1j} - (\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}) \frac{\partial(u_i^{(1)} + u_i^{(2)})}{\partial x_1} \right] n_j d\Gamma. \tag{A.3}$$

Expanding and rearranging terms gives

$$J^{(1+2)} = J^{(1)} + J^{(2)} + I^{(1,2)} \tag{A.4}$$

where $I^{(1,2)}$ is called the interaction integral for states 1 and 2

$$I^{(1,2)} = \int_{\Gamma} \left[W^{(1,2)} \delta_{1j} - \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} \right] n_j \, d\Gamma \tag{A.5}$$

where $W^{(1,2)}$ is the interaction strain energy

$$W^{(1,2)} = \sigma_{ij}^{(1)} \epsilon_{ij}^{(2)} = \sigma_{ij}^{(2)} \epsilon_{ij}^{(1)} \tag{A.6}$$

Writing equation (A.1) for the combined states gives after rearranging terms

$$J^{(1+2)} = J^{(1)} + J^{(2)} + \frac{2}{E^*} \left(K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right) \tag{A.7}$$

Equating (A.4) with (A.7) leads to the following relationship

$$I^{(1,2)} = \frac{2}{E^*} \left(K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right) \tag{A.8}$$

Making the judicious choice of state 2 as the pure Mode I asymptotic fields with $K_I^{(2)} = 1$, $K_{II}^{(2)} = 0$ gives the mode I stress intensity factor for state 1 in terms of the interaction integral

$$K_I^{(1)} = \frac{2}{E^*} I^{(1,\text{Mode } I)} \tag{A.9}$$

The mode II stress intensity factor can be determined in a similar fashion.

The contour integral (A.5) is not in a form best suited for finite element calculations. We therefore recast the integral into an equivalent domain form by multiplying the integrand by a sufficiently smooth weighting function $q(\boldsymbol{x})$ which takes a value of unity on an open set containing the crack tip and vanishes on an outer prescribed contour C_0. Then for each contour Γ as in Fig. A.1, assuming the crack faces are straight in the region A bounded by the contour C_0, the interaction integral may be

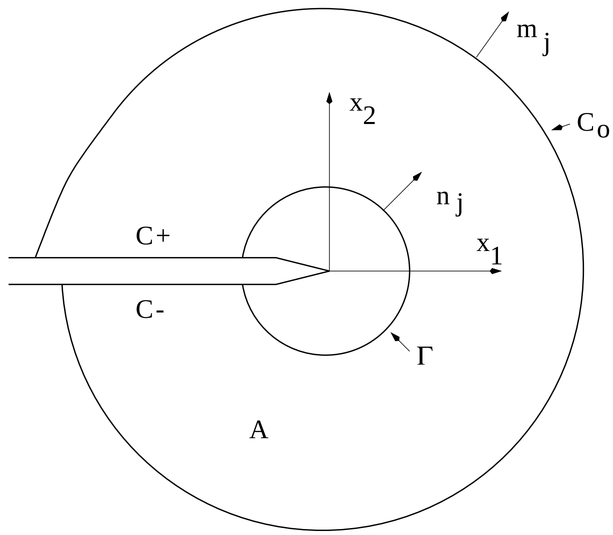

Figure A.1: Conventions at crack tip. Domain A is enclosed by Γ, C_+, C_-, and C_o. Unit normal $m_j = n_j$ on C_+, C_-, and C_o and $m_j = -n_j$ on Γ.

written as

$$\begin{aligned} I^{(1,2)} = & \int_C \left[\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right] q m_j \; dC \\ & - \int_{C_+ + C_-} \left[\sigma_{i2}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{i2}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} \right] q m_2 \; dC \end{aligned} \tag{A.10}$$

where the contour $C = \Gamma + C_+ + C_- + C_0$ and $\vec{m}$ is the unit outward normal. In deriving the above, we have used the relations $m_j = -n_j$ on Γ and $m_j = n_j$ on C_o, C_+ and C_-, and $m_1 = 0$ on C_+ and C_-.

Now using the divergence theorem on the closed integral over the contour C and passing to the limit as the contour Γ is shrunk to the crack tip, we arrive at the following equation for the interaction integral in domain form:

$$\begin{aligned} I^{(1,2)} = & \int_A \left[\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right] \frac{\partial q}{\partial x_j} \; dA - \\ & \int_{C_+ + C_-} \left[\sigma_{i2}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{i2}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} \right] q m_2 \; dC \end{aligned} \tag{A.11}$$

In deriving the above, we have used the knowledge that the expression $\sigma_{i2}^{(2)} m_2, \quad i = 1, 2$ vanishes on the crack faces as the auxiliary fields satisfy traction-free crack faces. We also note the importance of including the integral over C_+ and C_- in the case of contact between the crack faces. While the normal tractions which arise on the crack faces during contact do not contribute to the J-integral (the normal forces do no work during crack extension), they do contribute to the interaction integral. The actual normal forces do work in conjunction with the auxiliary displacements.

For the numerical evaluation of the above integral, the domain A is set from the collection of elements about the crack tip. In this paper, we first determine the characteristic length of an element touched by the crack tip and designate this quantity as h_{local}. For two dimensional analysis, this quantity is calculated as the square root of the element area. The domain A is then set to be all elements which have a node within a ball of radius r_d about the crack tip.

Fig. A.2 shows a typical set of elements for the domain A with the domain radius r_d taken to be twice the length h_{local}. Fig. A.3 shows the contour plot of the weight function q for these elements. The weight function q is taken to have a value of unity for all nodes within the ball r_d, and zero on the outer contour. The function is then easily interpolated within the elements using the nodal shape functions.

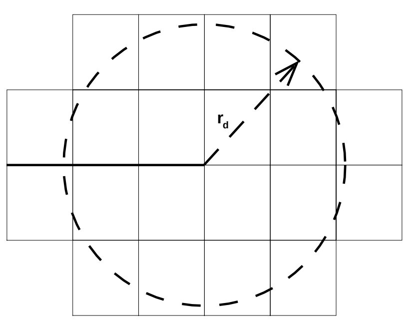

Figure A.2: Elements selected about the crack tip for the interaction integral.

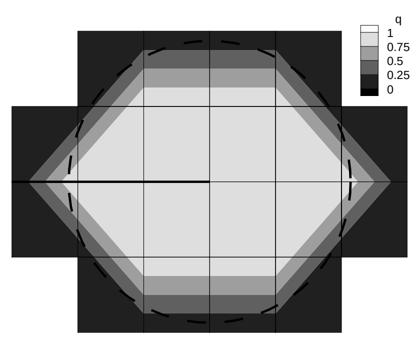

Figure A.3: Weight function q on the elements.

Appendix B

Derivation of Contact Update Equations

In this appendix, we derive the local update equations for each of the constitutive laws presented in Chapter 5. This involves using the constitutive law in conjunction with the search directions to calculate the quantities $(\boldsymbol{t}^I, \boldsymbol{w}^I)$ given $(\boldsymbol{t}^A, \boldsymbol{w}^A)$. These equations are developed for perfect contact, unilateral contact, and unilateral contact with friction in the following sections. The following development closely follows that of Champaney (1996), and provides some additional details.

B.1 Perfect Contact

We first recall the constitutive law for perfect contact, which states continuity in displacements across Γ_d as well as equilibrium of forces:

$$\boldsymbol{w}^+ = \boldsymbol{w}^-, \quad \boldsymbol{t}^+ = -\boldsymbol{t}^- \tag{B.1}$$

The other pertinent equations correspond to the search direction:

$$\boldsymbol{t}^{I+} - \boldsymbol{t}^{A+} = \boldsymbol{k}_0(\boldsymbol{w}^{I+} - \boldsymbol{w}^{A+}) \tag{B.2a}$$

$$\boldsymbol{t}^{I-} - \boldsymbol{t}^{A-} = \boldsymbol{k}_0(\boldsymbol{w}^{I-} - \boldsymbol{w}^{A-}) \tag{B.2b}$$

We recall that the constitutive laws only hold for those quantities in $\boldsymbol{I}_l$, namely $(\boldsymbol{t}^{I+}, \boldsymbol{t}^{I-})$ and $(\boldsymbol{w}^{I+}, \boldsymbol{w}^{I-})$.

Taking the sum of (B.2a) and (B.2b), and using (B.1) yields

$$-\boldsymbol{t}^{A+} - \boldsymbol{t}^{A-} = \boldsymbol{k}_0(2\boldsymbol{w}^{I+} - \boldsymbol{w}^{A+} - \boldsymbol{w}^{A-}) \tag{B.3}$$

which can be solved for $\boldsymbol{w}^{I+}$, giving

$$\boldsymbol{w}^{I+} = \frac{1}{2}\left\{(\boldsymbol{w}^{A+} + \boldsymbol{w}^{A-}) - \boldsymbol{k}_0^{-1}(\boldsymbol{t}^{A+} + \boldsymbol{t}^{A-})\right\} \tag{B.4}$$

In a similar fashion, we take the difference of (B.2a) and (B.2b) together with (B.1) to obtain

$$2\boldsymbol{t}^{I+} - \boldsymbol{t}^{A+} + \boldsymbol{t}^{A-} = \boldsymbol{k}_0(\boldsymbol{w}^{A-} - \boldsymbol{w}^{A+}) \tag{B.5}$$

which can be rewritten as

$$\boldsymbol{t}^{I+} = \frac{1}{2}\left\{(\boldsymbol{t}^{A+} - \boldsymbol{t}^{A-}) - \boldsymbol{k}_0(\boldsymbol{w}^{A+} - \boldsymbol{w}^{A-})\right\} \tag{B.6}$$

With the aid of (B.1), (B.4) and (B.6) provide the local update equations for perfect contact as

$$\boldsymbol{w}^{I+} = \boldsymbol{w}^{I-} = \frac{1}{2}\left\{(\boldsymbol{w}^{A+} + \boldsymbol{w}^{A-}) - \boldsymbol{k}_0^{-1}(\boldsymbol{t}^{A+} + \boldsymbol{t}^{A-})\right\} \tag{B.7a}$$

$$\boldsymbol{t}^{I+} = -\boldsymbol{t}^{I-} = \frac{1}{2}\left\{(\boldsymbol{t}^{A+} - \boldsymbol{t}^{A-}) - \boldsymbol{k}_0(\boldsymbol{w}^{A+} - \boldsymbol{w}^{A-})\right\} \tag{B.7b}$$

B.2 Unilateral Contact without Friction

We now consider the case of unilateral contact without friction. When the faces of the interface are in contact, there is no gap in the normal components of the displacements and we expect compressive tractions. When the faces are separated, the displacement gap is arbitrary and the tractions are zero. This is stated more formally as

$$(\boldsymbol{w}^{-} - \boldsymbol{w}^{+}) \cdot \boldsymbol{n} \geq 0 \tag{B.8a}$$

$$\boldsymbol{t}^{+} \cdot \boldsymbol{n} \leq 0, \quad \boldsymbol{t}^{-} \cdot \boldsymbol{n} \geq 0, \quad \boldsymbol{t}^{+} \cdot \boldsymbol{n} = -\boldsymbol{t}^{-} \cdot \boldsymbol{n} \tag{B.8b}$$

$$(\boldsymbol{t}^{+} \cdot \boldsymbol{n})(\boldsymbol{w}^{-} - \boldsymbol{w}^{+}) \cdot \boldsymbol{n} = 0 \tag{B.8c}$$

$$\boldsymbol{P}_T\boldsymbol{t}^{+} = \boldsymbol{P}_T\boldsymbol{t}^{-} = 0 \tag{B.8d}$$

where $\boldsymbol{n}$ is the unit normal vector to Γ_d^+, and $\boldsymbol{P}_T$ is the tangential projection operator.

It is also advantageous to recast the search equations (B.2) in terms of normal and tangential components. Taking the operator $\boldsymbol{k}_0$ to be the identity matrix multiplied by a constant k, we rewrite (B.2) as

$$(\boldsymbol{t}^{I+} - \boldsymbol{t}^{A+}) \cdot \boldsymbol{n} = k(\boldsymbol{w}^{I+} - \boldsymbol{w}^{A+}) \cdot \boldsymbol{n} \tag{B.9a}$$

$$(\boldsymbol{t}^{I-} - \boldsymbol{t}^{A-}) \cdot \boldsymbol{n} = k(\boldsymbol{w}^{I-} - \boldsymbol{w}^{A-}) \cdot \boldsymbol{n} \tag{B.9b}$$

$$\boldsymbol{P}_T(\boldsymbol{t}^{I+} - \boldsymbol{t}^{A+}) = k\boldsymbol{P}_T(\boldsymbol{w}^{I+} - \boldsymbol{w}^{A+}) \tag{B.9c}$$

$$\boldsymbol{P}_T(\boldsymbol{t}^{I-} - \boldsymbol{t}^{A-}) = k\boldsymbol{P}_T(\boldsymbol{w}^{I-} - \boldsymbol{w}^{A-}) \tag{B.9d}$$

We recall that the constitutive law applies to the quantities $(\boldsymbol{t}^{I+}, \boldsymbol{t}^{I-})$ and $(\boldsymbol{w}^{I+}, \boldsymbol{w}^{I-})$. For this constitutive law, we must first determine whether or not the two surfaces are in contact, and then proceed accordingly. In either situation, we have the following relations for the normal components. In contact:

$$(\boldsymbol{w}^{I-} - \boldsymbol{w}^{I+}) \cdot \boldsymbol{n} = 0, \quad \boldsymbol{t}^{I+} \cdot \boldsymbol{n} = -\boldsymbol{t}^{I-} \cdot \boldsymbol{n} \leq 0 \tag{B.10}$$

and in separation:

$$(\boldsymbol{w}^{I-} - \boldsymbol{w}^{I+}) \cdot \boldsymbol{n} > 0, \quad \boldsymbol{t}^{I+} \cdot \boldsymbol{n} = -\boldsymbol{t}^{I-} \cdot \boldsymbol{n} = 0 \tag{B.11}$$

Now consider the following contact indicator:

$$2C^I = (\boldsymbol{w}^{I-} - \boldsymbol{w}^{I+}) \cdot \boldsymbol{n} - \frac{1}{k}(\boldsymbol{t}^{I-} - \boldsymbol{t}^{I+}) \cdot \boldsymbol{n} \tag{B.12}$$

By examining (B.10) and (B.11) above, we see that the surfaces are in contact when $C^I \leq 0$ and are in separation when $C^I > 0$.

The expression for C^I above is not of much use, especially when the known quantities are $(\boldsymbol{t}^A, \boldsymbol{w}^A)$. By taking the difference of (B.9b) with (B.9a), we obtain

$$(\boldsymbol{t}^{I-} - \boldsymbol{t}^{I+}) \cdot \boldsymbol{n} - (\boldsymbol{t}^{A-} - \boldsymbol{t}^{A+}) \cdot \boldsymbol{n} = k(\boldsymbol{w}^{I-} - \boldsymbol{w}^{I+}) \cdot \boldsymbol{n} - k(\boldsymbol{w}^{A-} - \boldsymbol{w}^{A+}) \cdot \boldsymbol{n} \tag{B.13}$$

which is rearranged easily enough as

$$(\boldsymbol{w}^{I-} - \boldsymbol{w}^{I+}) \cdot \boldsymbol{n} - \frac{1}{k}(\boldsymbol{t}^{I-} - \boldsymbol{t}^{I+}) \cdot \boldsymbol{n} = (\boldsymbol{w}^{A-} - \boldsymbol{w}^{A+}) \cdot \boldsymbol{n} - \frac{1}{k}(\boldsymbol{t}^{A-} - \boldsymbol{t}^{A+}) \cdot \boldsymbol{n} \tag{B.14}$$

such that C_I can be determined equivalently from

$$2C_I = (\boldsymbol{w}^{A-} - \boldsymbol{w}^{A+}) \cdot \boldsymbol{n} - \frac{1}{k}(\boldsymbol{t}^{A-} - \boldsymbol{t}^{A+}) \cdot \boldsymbol{n} \tag{B.15}$$

If separation is determined ($C_I > 0$), all components of the tractions on Γ_d are zero, and so from (B.2) we recover

$$\boldsymbol{w}^{I+} = \boldsymbol{w}^{A+} - \boldsymbol{k}_0^{-1}\boldsymbol{t}^{A+} \tag{B.16a}$$

$$\boldsymbol{w}^{I-} = \boldsymbol{w}^{A-} - \boldsymbol{k}_0^{-1}\boldsymbol{t}^{A-} \tag{B.16b}$$

$$\boldsymbol{t}^{I+} = \boldsymbol{t}^{I-} = 0 \tag{B.16c}$$

If contact is determined ($C_I \leq 0$), the update equations for the normal components of the tractions $\boldsymbol{t}^I$ and displacements $\boldsymbol{w}^I$ are determined by taking the difference and sum of equations (B.9b) and (B.9a). The difference was already determined in the expression for C_I. Using the expressions (B.10) and (B.13) we can then write

$$\boldsymbol{t}^{I+} \cdot \boldsymbol{n} = -\boldsymbol{t}^{I-} \cdot \boldsymbol{n} = \frac{1}{2}\left\{k(\boldsymbol{w}^{A-} - \boldsymbol{w}^{A+}) \cdot \boldsymbol{n} - (\boldsymbol{t}^{A-} - \boldsymbol{t}^{A+}) \cdot \boldsymbol{n})\right\} = kC_I \tag{B.17}$$

giving the update equations for the normal components of the tractions. Taking the sum of (B.9a) and (B.9b) yields

$$(\boldsymbol{t}^{I-} + \boldsymbol{t}^{I+}) \cdot \boldsymbol{n} - (\boldsymbol{t}^{A-} + \boldsymbol{t}^{A+}) \cdot \boldsymbol{n} = k(\boldsymbol{w}^{I-} + \boldsymbol{w}^{I+}) \cdot \boldsymbol{n} - k(\boldsymbol{w}^{A-} + \boldsymbol{w}^{A+}) \cdot \boldsymbol{n} \tag{B.18}$$

Using the relations for contact (B.10), we can then write

$$\boldsymbol{w}^{I+} \cdot \boldsymbol{n} = \boldsymbol{w}^{I-} \cdot \boldsymbol{n} = \frac{1}{2}\left\{(\boldsymbol{w}^{A+} + \boldsymbol{w}^{A-}) - \frac{1}{k}(\boldsymbol{t}^{A+} + \boldsymbol{t}^{A-})\right\} \cdot \boldsymbol{n} \tag{B.19}$$

which are the update equations for the normal components of the displacements.

To determine the tangential components of the interfacial displacements and tractions, we begin by recalling that for frictionless contact the tangential tractions are zero. By making the appropriate substitutions into the tangential search equations (B.9c) and (B.9d), we then have

$$\boldsymbol{P}_T\boldsymbol{t}^{I+} = -\boldsymbol{P}_T\boldsymbol{t}^{I-} = 0 \tag{B.20a}$$

$$\boldsymbol{P}_T\boldsymbol{w}^{I+} = \boldsymbol{P}_T\boldsymbol{w}^{A+} + \frac{1}{k}\boldsymbol{P}_T(\boldsymbol{t}^{I+} - \boldsymbol{t}^{A+}) \tag{B.20b}$$

$$\boldsymbol{P}_T\boldsymbol{w}^{I-} = \boldsymbol{P}_T\boldsymbol{w}^{A-} + \frac{1}{k}\boldsymbol{P}_T(\boldsymbol{t}^{I-} - \boldsymbol{t}^{A-}) \tag{B.20c}$$

B.3 Unilateral Contact with Friction

We now consider the case of contact with friction on the interface, in which the tangential tractions are not necessarily zero. When the friction is idealized using a

Coulomb law, the maximum frictional force supported on the interface is

$$g = \mu|\boldsymbol{t} \cdot \boldsymbol{n}| \tag{B.21}$$

where μ is the coefficient of friction. The constitutive law is restated as

$$(\boldsymbol{w}^- - \boldsymbol{w}^+) \cdot \boldsymbol{n} \geq 0 \tag{B.22a}$$
$$\boldsymbol{t}^+ \cdot \boldsymbol{n} \leq 0, \quad \boldsymbol{t}^- \cdot \boldsymbol{n} \geq 0, \quad \boldsymbol{t}^+ \cdot \boldsymbol{n} = -\boldsymbol{t}^- \cdot \boldsymbol{n} \tag{B.22b}$$
$$(\boldsymbol{t}^+ \cdot \boldsymbol{n})(\boldsymbol{w}^- - \boldsymbol{w}^+) \cdot \boldsymbol{n} = 0 \tag{B.22c}$$
$$||\boldsymbol{P}_T\boldsymbol{t}|| \leq g \tag{B.22d}$$

Additional equations involve the tangential components of the displacement and correspond to whether there is 'stick' or 'slip' on the interface, i.e.

$$\boldsymbol{P}_T\boldsymbol{w}^- = \boldsymbol{P}_T\boldsymbol{w}^+ \qquad \text{if} \quad ||\boldsymbol{P}_T\boldsymbol{t}|| \leq g \quad (\text{'stick'}) \tag{B.23a}$$
$$\boldsymbol{P}_T(\boldsymbol{w}^- - \boldsymbol{w}^+) = -\lambda\boldsymbol{P}_T\boldsymbol{t}^+ \qquad \text{if} \quad ||\boldsymbol{P}_T\boldsymbol{t}|| = g \quad (\text{'slip'}) \tag{B.23b}$$

where $\lambda > 0$ is some positive constant.

The update equations for this constitutive law begin much in the same way as for the case of unilateral contact without friction. If there is separation ($C_I > 0$), the local quantities $(\boldsymbol{t}^I, \boldsymbol{w}^I)$ are determined from (B.16). If there is contact, the normal components are given by (B.17) and (B.19). To update the tangential components, we must first determine whether slip or stick conditions exist on the interface.

Consider the following vectorial indicator $\boldsymbol{G}^I$

$$\boldsymbol{G}^I = \frac{1}{2}\left\{k\boldsymbol{P}_T(\boldsymbol{w}^{I-} - \boldsymbol{w}^{I+}) - \boldsymbol{P}_T(\boldsymbol{t}^{I-} - \boldsymbol{t}^{I+})\right\} \tag{B.24}$$

and its norm $||\boldsymbol{G}^I||$. When there is 'stick', the tangential displacements are zero and from (B.23a) we see that

$$||G^I|| = ||\boldsymbol{P}_T\boldsymbol{t}^{I+}|| \leq g \tag{B.25}$$

where we have used $||\boldsymbol{P}_T\boldsymbol{t}^{I+}|| = ||\boldsymbol{P}_T\boldsymbol{t}^{I-}||$. When there is 'slip' on the interface, we have

$$||G^I|| = \frac{k\lambda}{2}||\boldsymbol{P}_T\boldsymbol{t}^{I+}|| + ||\boldsymbol{P}_T\boldsymbol{t}^{I+}|| > g \tag{B.26}$$

and so $||G^I|| > g$ indicates slip while $||G^I|| \leq g$ indicates stick.

As with C^I, the expression for G^I in terms of $(\boldsymbol{t}^I, \boldsymbol{w}^I)$ is not very useful. In an analagous manner, we may manipulate the search equations for the tangential components to rewrite (B.24) as

$$\boldsymbol{G}^I = \frac{1}{2}\left\{k\boldsymbol{P}_T(\boldsymbol{w}^{A-} - \boldsymbol{w}^{A+}) - \boldsymbol{P}_T(\boldsymbol{t}^{A-} - \boldsymbol{t}^{A+})\right\} \tag{B.27}$$

When $||G^I|| \leq g$, we effectively recover perfect contact on the interface. The local update for the tangential components is then obtained by considering (B.7) in tangential form. This gives

$$\boldsymbol{P}_T\boldsymbol{t}^{I+} = -\boldsymbol{P}_T\boldsymbol{t}^{I-} = \boldsymbol{G}^I \tag{B.28a}$$

$$\boldsymbol{P}_T\boldsymbol{w}^{I+} = \boldsymbol{P}_T\boldsymbol{w}^{I-} = \boldsymbol{w}^{A+} + \boldsymbol{k}_0^{-1}\boldsymbol{P}_T(\boldsymbol{t}^{I+} - \boldsymbol{t}^{A+}) \tag{B.28b}$$

In the case of slip on the interface, we know that the magnitude of the tangential forces is equal to g. From the previous expression, we see that the direction is indicated by $\boldsymbol{G}^I$. The local update for the tractions is then

$$\boldsymbol{P}_T\boldsymbol{t}^{I+} = -\boldsymbol{P}_T\boldsymbol{t}^{I-} = g\frac{\boldsymbol{G}^I}{||\boldsymbol{G}^I||} \tag{B.29}$$

With the tractions determined, the tangential components of the displacements are obtained with the search equations (B.9c) and (B.9d):

$$\boldsymbol{P}_T\boldsymbol{w}^{I+} = \boldsymbol{P}_T\boldsymbol{w}^{A+} + \frac{1}{k}\boldsymbol{P}_T(\boldsymbol{t}^{I+} - \boldsymbol{t}^{A+}) \tag{B.30a}$$

$$\boldsymbol{P}_T\boldsymbol{w}^{I-} = \boldsymbol{P}_T\boldsymbol{w}^{A-} + \frac{1}{k}\boldsymbol{P}_T(\boldsymbol{t}^{I-} - \boldsymbol{t}^{A-}) \tag{B.30b}$$

This completes the update equations for frictional contact.