

# Chapter 1

## Introduction

This chapter provides an introduction to discontinuous enrichment in the finite element framework. Section 1.1 reviews some of the advancements made in element and approximation technology in the past decade. The manner in which the present perspective of this thesis was reached is also discussed. The methods developed in this work find natural applications in the field of fracture mechanics, and the pertinent issues are reviewed in Section 1.2. The following section discusses the modeling of nonlinear constitutive laws on an interface, and suggests the advantages of representing the interface with discontinuous enrichment. Finally, Section 1.4 provides an outline of the remainder of the thesis.

### 1.1 Element and Approximation Technology

The finite element method (FEM) is one of the most popular numerical methods for constructing approximate solutions to differential equations. The FEM is used by scientists and engineers to investigate the dynamic failure of structures, turbulent

flow around an airfoil, and thermal loads on electronic micro-chips, just to name a few applications. While the finite element method is well developed and robust, it is not particularly well suited to model evolving discontinuities. As a specific instance of a Galerkin method, it relies on an element structure to construct an approximating space. The construction of a discontinuous space with finite elements necessitates alignment of the element topology with the geometry of the discontinuity. This in turn entails a regeneration of the mesh as the discontinuity evolves, resulting in projection errors and a significant computational cost.

Over the course of the past several years, much attention has been focused on developing new approximations in the Galerkin framework which do not require a mesh for their construction. These so-called ‘meshfree’ or ‘meshless’ methods construct approximations from a set of nodal data and associated weight functions with compact support on the domain. Some of the concepts in meshless methods can be traced to the Smoothed Particle Hydrodynamics (SPH) method (Gingold and Monaghan, 1977) , which implements a kernel function to construct locally conservative discrete approximations. For an arbitrary arrangement of particles (grid points), the consistency of the approximation depends on the selection of the smoothing ‘kernel’ or weight function. Unfortunately, convergence is only met if the support of the kernel function is allowed to grow with respect to nodal spacing, rendering the method fairly impractical for most engineering applications (Mas-Gallic and Raviart, 1987). Several modifications have recently been made to the standard method, with applications to the brittle fracture and large deformation of solids (Randles and Libersky, 1996).

A parallel line of development concerns the diffuse-element method (DEM) (Nayroles, Touzot, and Villon, 1992) and element-free Galerkin (EFG) method (Belytschko, Lu, and Gu, 1994). For these meshless methods an approximation is constructed from

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a set of nodes and a moving-least-squares technique developed for curve and surface fitting (Lancaster and Salkauskas, 1981). For certain nodal arrangements and choice of weight functions, Belytschko et al. (1996) showed that the basis functions of the EFG method were equivalent to those of the RKPM method (Liu, Jun, and Zhang, 1995). The RKPM formulation demonstrated that the SPH kernel could be used to construct a basis for an approximating space in the Galerkin framework, with the proviso that certain corrections be made to ensure the linear precision of the resulting approximation. The correction requires solving a small system of equations at each point in the domain, but convergence is achieved without expanding the support of the weight functions. Both the EFG and RKPM methods were later generalized as signed partition of unity approximations by Melenk and Babuška (1996).

Meshless methods have been applied to problems in applied mechanics for which the use of a traditional finite element mesh presents significant difficulties. These include the modeling of quasi-static and dynamic fracture in solids by both the EFG and SPH methods, and the modeling of fluid flow and large deformation in solids by the RKPM. In each case, the construction of an approximation which does not rely on an element topology is seen to be a significant advantage. However, meshless methods require considerably greater computational resources than the finite element method. The additional cost is associated with the construction of the basis functions and the additional quadrature required for the accurate construction of the bilinear form. Difficulties associated with the satisfaction of essential boundary conditions present further obstacles.

Both quasi-static and dynamic fracture are local phenomena, in that the crack geometry is confined to a relatively small portion of the domain. An alternative to meshless approximations concerns the development of finite elements which are

capable of representing inter-element discontinuities. These elements are then used locally about the crack geometry to represent the arbitrary discontinuity. This work began in the context of modeling localization zones (Belytschko, Fish, and Engleman, 1988), (Simo, Oliver, and Armero, 1993), and continues with the more recent work of Armero and Garikipati (1996) and Oliver (1996). The foregoing methods all fall into the assumed strain category, in which the displacement jump is represented as a piecewise constant field across element boundaries.

Many of the advantages of both meshless and assumed strain methods can be realized in the partition of unity framework, where local enrichment functions are incorporated into the approximation in a straightforward fashion (Melenk and Babuška, 1996). This framework is viewed to be more flexible than the assumed strain approach, in that it involves a direct modification to the displacement approximation. The utility of the method has found application in solving the scalar Laplacian problem for domains with re-entrant corners (Strouboulis, Babuška, and Copps, 1998). The concept of incorporating local functions into an approximation is not new, see for example hybrid methods (Jirousek and Wroblewski, 1996). The partition of unity framework is distinguished by the multiplication of the local enrichment functions by nodal shape functions, which results in a conforming approximation. The application of this idea to capture a specific frequency band in dynamics can also be found in Liu, Zhang, and Ramirez (1991), where the approximation over the entire domain was modified.

Belytschko and Black (1999) adopted the partition of unity concept to model crack growth, by locally enriching a finite element approximation with the exact near tip crack fields. A key feature of this work was the incorporation of discontinuous enrichment functions, and the use of a mapping procedure to model arbitrary dis-

continuities. The incorporation of the near-tip fields also provided for accurate stress intensity factor calculations on relatively coarse meshes. Unfortunately, the mapping technique is awkward for long cracks, and the asymptotic near-tip functions are not the best choice to represent the displacement discontinuity far from the crack tip. As a result, some level of remeshing is implemented as the crack grows.

In this thesis, the work of Belytschko and Black (1999) is extended by incorporating a discontinuous field across the crack faces away from the crack tip. The addition of a discontinuous field allows for the entire crack geometry to be modeled independently of the mesh, and completely avoids the need to remesh as the crack grows. A specific set of rules for selecting enriched nodes from the relationship between the crack geometry and the finite element mesh is developed. These guidelines are implemented in conjunction with a modified integration scheme to construct the bilinear form. Much of the work developed in this thesis concerning discontinuous enrichment has been submitted for publication in Moës, Dolbow, and Belytschko (1999) and Dolbow, Moës, and Belytschko (1999).

## 1.2 Computational Fracture Mechanics

This section compares the present approach to previous work with finite elements in the area of fracture mechanics. In the finite element literature, several different approaches to model cracks and crack growth can be found. Consider for example, the hybrid elements developed by Tong, Pian, and Lasry (1973), and the more recent local-mesh replacement method (ALMR) developed by Rashid (1998). The enrichment strategy developed in this thesis shares features with both methods, in that the singular crack tip fields are incorporated and the approximation changes locally

as the crack propagates. It also possesses distinct advantages over each method. In comparison to hybrid formulations, the incorporation of local enrichment in the partition of unity framework results in a conforming approximate space. In comparison to the ALMR, an update of the element topology is not required behind the crack front as it propagates through the mesh.

Of critical importance in computational fracture mechanics is the determination of the parameters which characterize the stress and displacement fields in the vicinity of a crack tip. If the stress intensity factor exceeds the critical value of the material, crack growth and ultimately structural failure are possible. Several strategies have been developed to extract mixed-mode intensity factors using contour integrals derived from conservation laws. Moran and Shih (1987) cast the essential issues in the general framework of deriving domain integrals from momentum and energy-balance. The equivalent domain integrals are viewed to be better suited for finite element calculations, as the same quadrature points used for the construction of the bilinear form can be used to evaluate the domain integrals.

Many of the challenges associated with computational fracture mechanics come to light when modeling the fracture of Mindlin-Reissner plates. In this context, the main considerations involve the extraction of the mixed-mode intensity factors as well as the use of element formulations which do not exhibit shear locking. Somewhat surprisingly, these two issues have been investigated quite independently in the past. Parallel to the work of Moran and Shih (1987), derivations of crack contour integrals from conservation laws can be found in the literature on plate fracture (Sosa and Herrmann, 1989). Unfortunately, these contour integrals have not been cast in domain form, and are often inadequate for extracting the desired quantities. The modeling of plate fracture in the Mindlin-Reissner framework is further complicated

by the phenomenon of shear locking. For very thin plates, the imposition of the Kirchhoff constraint on the finite element subspace results in a degradation in accuracy for standard elements. While several different element formulations have been developed which avoid shear locking, they have not been implemented to study fracture problems.

In this thesis, both issues are examined by enriching an element which does not exhibit shear locking with both discontinuous and near tip fields. The development of a domain form of the interaction integral in the context of Mindlin-Reissner plate fracture facilitates the evaluation of the resulting formulation. The derivation shares many common features with Gosz, Dolbow, and Moran (1998), in that the inability of the auxiliary fields to satisfy the governing equations is a key consideration. This development allows for the extraction of mixed-mode moment and shear force intensity factors for plates in a straightforward fashion.

### 1.3 Modeling Evolving Interfaces

The use of discontinuous enrichment results in the generation of an interface which is independent of the finite element mesh. As a result, the method offers some unique opportunities to circumvent some of the traditional difficulties associated with the finite element modeling of nonlinear constitutive laws on interfaces.

The modeling of a nonlinear constitutive law at the interface between two surfaces is of practical importance for many engineering applications. Consider for example the case of unilateral contact between two components. Several methods are available to enforce the contact conditions at the interface in the finite element context, including penalty formulations. Penalty formulations are attractive on several levels;

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the introduction of a penalty parameter to enforce the local constraints is relatively simple, and the approach does not introduce additional unknowns. Unfortunately, the incorporation of the penalty parameter is equivalent to solving a different model of the boundary value problem, such as when a stiff spring is placed on the contact interface. As the spring stiffness is increased, the system of equations becomes poorly conditioned, while the contact conditions are enforced exactly only in the limit of an infinitely stiff spring.

In this work, the iterative resolution scheme adopted to satisfy the nonlinear constitutive law on the interface is borrowed from the LATIN method (Ladevèze, 1998). The method is quite general, in that it has been used to solve traditional problems in plasticity and viscoplasticity. In terms of its application to contact conditions on an interface, it shares similar features with the method of augmented Lagrangians as developed by Alart and Curnier (1991) and Simo and Laursen (1992). An estimate for the tractions on the interface is used to solve the global problem, which in turn generates a better approximation for the local interfacial tractions. The iterative strategy and discontinuous enrichment form a natural pair, in that it is straightforward to consider an arbitrary constitutive law on an arbitrary interface geometry.

## 1.4 Outline

An outline of the remainder of this thesis is as follows. Chapter 2 examines the solution to a one-dimensional model problem for various partition of unity methods. Emphasis is placed on the construction of an approximation and the numerical integration of the Galerkin weak form for a problem with local character. Chapter 3 introduces the concept of discontinuous enrichment in the context of modeling

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cracks and crack growth. A detailed description of the construction of the enriched approximation from the interaction of the crack and mesh geometries is provided. The extension of the technique to model plate fracture is examined in Chapter 4. In addition to developing the appropriate near-tip enrichment in the framework of Mindlin-Reissner plate theory, a domain form of the interaction integral is derived to extract the mixed-mode moment and shear force intensity factors. The use of discontinuous enrichment to model an interface with nonlinear constitutive laws is presented in Chapter 5. The combination of a well established incremental method with discontinuous enrichment is shown to provide a straightforward manner to model crack growth with contact on the crack faces. Chapter 6 presents several simulations of crack growth which take full advantage of the techniques developed in the thesis. Conclusions and future work are stated in Chapter 7. The derivation of the domain form of the interaction integral for two-dimensional elasticity, and the update schemes used for the contact algorithm are provided in Appendices A and B, respectively.